

JENSEN'S INEQUALITY FOR A CONVEX VECTOR-VALUED FUNCTION
ON AN INFINITE-DIMENSIONAL SPACE.*

by

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Abstract.

Jensen's inequality $f(EX) \leq Ef(X)$ for the expectation of a convex function of a random variable is extended to a generalized class of convex functions f whose domain and range are subsets of (possibly) infinite-dimensional linear topological spaces. Convexity of f is defined with respect to closed cone partial orderings, or more general binary relations, on the range of f . Two different methods of proof are given, one based on geometric properties of convex sets and the other based on the Strong Law of Large Numbers. Various conditions under which Jensen's inequality becomes strict are studied.

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Jensen's inequality, strict inequality, convex vector-valued function, partial ordering, closed cone ordering, linear topological space, dual space, convex set, supporting hyperplane, separating hyperplane, Strong Law of Large Numbers, Banach space, Frechet space, W -convexity, W -continuity.

1. Introduction.

Jensen's inequality for the expectation of a convex real-valued function of several real variables is as follows:

Proposition 1.1.

Let f be a convex function defined on a convex subset C of n -dimensional Euclidean space R^n and let $X = (X_1, \dots, X_n)$ be an integrable random vector such that $P[X \in C] = 1$. Then $EX \in C$, $Ef(X)$ exists, and

$$(1.1) \quad f(EX) \leq Ef(X).$$

Furthermore, if f is strictly convex and the distribution of X is not concentrated at a single point, then strict inequality holds in (1.1).

This well-known result has many applications in areas such as probability and statistics (cf. [6], [11]). The fact that EX lies in C is proved most easily by induction on the dimension n , while the inequality (1.1) follows from the existence of supporting hyperplanes for a convex set in R^{n+1} ; see Ferguson [6], p. 76 for details.

Jensen's inequality can be extended to convex vector-valued functions $f: C \rightarrow R^k$ if the convexity of R is defined with respect to a certain type of partial ordering, called a closed cone ordering (section 2), on its range in R^k . As an example, let C^p be the class of all $p \times p$ symmetric positive definite matrices. For s_1, s_2 in C^p define the partial ordering $s_1 \leq s_2$ to mean that $s_2 - s_1$ is positive semidefinite. Then the vector-valued function $f: C^p \rightarrow C^p$ defined by $f(s) = s^{-1}$ is convex with respect to this ordering. If S is a random positive definite matrix which is component-wise integrable, then Proposition 1.1 can be applied ([7], [15]) to show that $E(S^{-1}) - (ES)^{-1}$ is positive semidefinite, i. e.,

$$(1.2) \quad f(ES) \leq Ef(S),$$

a generalized version of (1.1). Here the partial ordering \leq is a closed cone ordering since $\{s | 0 \leq s\}$, the set of all positive semidefinite $p \times p$ matrices, is a closed convex cone. Notice that the ordering \leq is determined by a collection of linear functionals on C^p , since $s_1 \leq s_2$ iff $t's_1t \leq t's_2t$ for all vectors $t: p \times 1$.

Consider now a convex real-valued function f defined on a convex subset C of an infinite-dimensional linear topological space \mathcal{X} . Let X be an \mathcal{X} -valued random variable defined on a probability space (Ω, \mathcal{G}, P) with range in C , and assume the Pettis integral EX exists (see section 3). In this case Jensen's inequality (1.1) no longer holds in general. For example, let $\mathcal{X} = R^\infty$ (the space of all infinite sequences of real numbers with the topology of pointwise convergence) and let C be the set of all nonnegative sequences. Let $\{X_n\}$ be a sequence of nonnegative integrable real-valued random variables defined on (Ω, \mathcal{G}, P) and let $X : \Omega \rightarrow C$ be given by

$$X(\omega) = (X_1(\omega), X_2(\omega), \dots).$$

Define the real-valued convex function f on C by

$$(1.3) \quad f(x_1, x_2, \dots) = \begin{cases} \limsup x_n & \text{if } \limsup x_n < \infty \\ 0 & \text{if } \limsup x_n = \infty. \end{cases}$$

Jensen's inequality (1.1), if it were true in this case, would imply Fatou's Lemma, i. e.,

$$(1.4) \quad \limsup EX_n \leq E(\limsup X_n).$$

It is well-known, however, that (1.4) is not true without further assumptions.

(Let (Ω, G, P) be the unit interval $(0, 1)$ with Lebesgue measure and consider $X_n(\omega) \equiv nI_{(0, n^{-1})}(\omega)$.) In this example, Jensen's inequality fails because the function f , although convex, is nonetheless quite irregular on C . This is in marked contrast to the finite-dimensional case, where a real-valued convex function must be continuous on the interior of its domain. Therefore, to extend Jensen's inequality to an infinite-dimensional space, some continuity assumption must be imposed on the convex function f . (see Theorems 3.2-3.10 and 4.1). (The example of this paragraph is discussed further after Theorem 3.6).

In this paper we obtain generalizations of Jensen's inequality of the forms (1.1) and (1.2) for a convex function f whose domain and range are subsets of (possibly) infinite-dimensional linear spaces. Convexity of f is taken with respect to certain closed cone orderings (discussed in section 2) or more general binary relations (section 4). The generalized Jensen's inequalities in section 3 are obtained by utilizing geometrical properties of convex sets in a linear topological space, while in section 4 a short proof is based on the Strong Law of Large Numbers in a Banach space. This latter provides, incidentally, a very short proof of (1.1) for the finite-dimensional case. Conditions for strict inequality are carefully examined in section 3. This last topic is complicated by the fact that there are several different ways to define strict inequality with respect to a partial ordering.

The following notations and conventions are used throughout. All linear topological spaces (LTS) considered are understood to be real and Hausdorff. If \mathcal{X} is a LTS, \mathcal{X}^* denotes the dual space of all real-valued continuous linear functionals on \mathcal{X} , and $\mathcal{B}(\mathcal{X})$ denotes the Borel σ -field

generated by all open subsets of \mathcal{X} . The zero element of a LTS is denoted by φ , real scalars by α (with or without subscripts) and the real line by \mathbb{R} . The interior, closure, and boundary of a set A are denoted by A^0 , \bar{A} , and ∂A respectively.

2. Partial orderings in a linear topological space and convex vector-valued functions.

In this section we discuss the partial orderings used to define convexity of a function f whose range is a subset of a LTS. Let \mathcal{U} be a LTS and \leq a partial ordering on \mathcal{U} , i.e., \leq is reflexive ($y \leq y$) and transitive ($y_1 \leq y_2 \leq y_3 \Rightarrow y_1 \leq y_3$). We say \leq is a closed cone ordering on \mathcal{U} if it satisfies two additional properties:

$$(i) \quad y_1 \leq y_2, y_3 \in \mathcal{U}, 0 \leq \alpha \Rightarrow \alpha(y_1 + y_3) \leq \alpha(y_2 + y_3);$$

$$(ii) \quad \text{if } \{y_i\} \text{ and } \{z_i\} \text{ are convergent nets in } \mathcal{U} \text{ such that } y_i \leq z_i \text{ for all } i, \text{ then } \lim y_i \leq \lim z_i.$$

The ordering \leq is called antisymmetric if $y \leq z, z \leq y \Rightarrow y = z$.

A set $K \subset \mathcal{U}$ is a closed convex cone if K is closed, convex, and if $y \in K, \alpha \geq 0 \Rightarrow \alpha y \in K$ (hence $\varphi \in K$). We say K is pointed if $K \cap (-K) = \{\varphi\}$. Each closed convex cone K determines a closed cone ordering \leq_K defined as follows: $y \leq_K z$ iff $z - y \in K$. Conversely if \leq is a closed cone ordering, the set $K = \{y \mid y \in \mathcal{U}, \varphi \leq y\}$ is a closed convex cone and the induced ordering \leq_K coincides with the original \leq . Thus there is a 1 - 1 correspondence between closed cone orderings and closed convex cones. Furthermore K is pointed iff \leq_K is antisymmetric.

If T is an arbitrary subset of \mathcal{U}^* the set

$$K_T = \bigcap_{t \in T} \{y \mid t(y) \geq 0\}$$

is a closed convex cone and therefore determines a closed cone ordering

$\leq_T \equiv \leq_{K_T}$. We call \leq_T the component-wise ordering determined by T .

The mapping $T \rightarrow K_T$ need not be 1 - 1, and not every closed cone ordering

need be a component-wise ordering. It follows from well-known separation

theorems (e.g. [5], Proposition 21.17), however, that if K is a closed

convex cone with $K^\circ \neq \emptyset$ then there exists $T \subset \mathcal{U}^*$ such that $K = K_T$.

Furthermore if \mathcal{U} is locally convex then the assumption $K^\circ \neq \emptyset$ can be

dropped ([5], Corollary 21.15), so in this case the set of closed cone orderings

and the set of component-wise orderings coincide. We say T is total on \mathcal{U}

if $t(y) = 0 \quad \forall t \in T \Rightarrow y = \emptyset$. T is total iff K_T is pointed.

For any partial ordering \leq we write $y < z$ to indicate that $y \leq z$

and $y \neq z$. If $\leq = \leq_T$ is a component-wise ordering we write $y \ll z$ to

mean that $t(y) < t(z) \quad \forall t \in T$. Clearly $y \ll z \Rightarrow y < z$.

Now let C be a convex subset of a LTS \mathcal{X} , $f: C \rightarrow \mathcal{U}$ a vector-valued function, and \leq a closed cone ordering on \mathcal{U} . We say f is convex with respect to \leq if

$$f(\alpha u + (1 - \alpha)v) \leq \alpha f(u) + (1 - \alpha) f(v)$$

whenever $u, v \in C$ and $0 \leq \alpha \leq 1$. It easily follows that for each integer $n \geq 2$,

$$f\left(\sum_{k=1}^n \alpha_k x_k\right) \leq \sum_{k=1}^n \alpha_k f(x_k)$$

whenever each $x_k \in C$ and $\alpha_k \geq 0$ with $\sum \alpha_k = 1$. The function f is strictly convex with respect to \leq if

$$f(\alpha u + (1 - \alpha)v) < \alpha f(u) + (1 - \alpha) f(v)$$

whenever $u, v \in C$, $u \neq v$, $0 < \alpha < 1$. If \leq is in fact a component-wise ordering, f is strictly component-wise convex with respect to \leq if

$$f(\alpha u + (1 - \alpha) v) \ll \alpha f(u) + (1 - \alpha) f(v)$$

whenever $u, v \in C$, $u \neq v$, $0 < \alpha < 1$.

3. Jensen's inequality for Pettis integrable functions.

The generalizations of Jensen's inequality obtained in this section are valid under the minimal assumption of Pettis integrability. We take \mathcal{X} to be a LTS and (Ω, \mathcal{G}, P) a probability space. A mapping $X: \Omega \rightarrow \mathcal{X}$ is Pettis integrable [9], [14] with respect to (Ω, \mathcal{G}, P) if (a) X is weakly measurable, i.e., $x^*(X)$ is Borel measurable $\forall x^* \in \mathcal{X}^*$; (b) $\int x^*(X) dP$ exists $\forall x^* \in \mathcal{X}^*$; and (c) there exists $\hat{x} \in \mathcal{X}$ such that $x^*(\hat{x}) = \int x^*(X) dP \forall x^* \in \mathcal{X}^*$. We denote \hat{x} by EX . If \mathcal{X}^* is total on \mathcal{X} then the Pettis integral is uniquely determined if it exists; in particular this is the case if \mathcal{X} is locally convex.

A closed hyperplane $H = \{x | x^* = \alpha\}$ is a supporting hyperplane of a set $A \subset \mathcal{X}$ if $x^*(A) \leq \alpha$ (or $x^*(A) \geq \alpha$) and $H \cap A \neq \emptyset$. Theorem 3.1 is an easy extension of a result in Bourbaki ([3], Théorème 1, Chap. IV, § 6).

Theorem 3.1

Suppose X is Pettis integrable. Let C be a convex subset of \mathcal{X} such that the range $X(\Omega) \subseteq C$.

(i) If either (a) $C^0 \neq \emptyset$ or (b) \mathcal{X} is locally convex, then $EX \in \overline{C}$.

(ii) If $C^0 \neq \emptyset$ and if $P[x^*(X) = \alpha] < 1$ whenever $\{x | x^*(x) = \alpha\}$ is a supporting hyperplane of \overline{C} , then $EX \in C^0$.

Proof:

(i) If either (a) or (b) hold then \bar{C} can be expressed as an intersection of closed halfspaces, i.e., $\bar{C} = \bigcap_i \{x | x_i^*(x) \leq \alpha_i\}$ ([5], Corollary 21.14 and Proposition 21.17). For each i , $P[x_i^*(X) \leq \alpha_i] = 1$ since $X(\Omega) \subseteq C$, so $\alpha_i \geq Ex_i^*(X) = x_i(EX)$, hence $EX \in \bar{C}$.

(ii) Suppose $EX \in \partial X$. Since \bar{C} is a convex body we can assume that $x_i^*(EX) = \alpha_i$ for some i ([5], Proposition 21.17). However, the hypothesis implies that $x_i^*(EX) < \alpha_i$, a contradiction, so $EX \in C^O$.

Remark 3.1.

If $\{\omega | X(\omega) \in C^O\}$ contains a set of positive P -measure then the hypotheses of Theorem 3.1 (ii) are satisfied, since $C^O \cap H = \emptyset$ for every supporting hyperplane H of \bar{C} .

Remark 3.2.

In Theorem 3.1 (i) it can happen that $EX \notin C$. For example let $X = R^\infty$ and let C be the subset consisting of all sequences having only finitely many nonzero elements. Then X is locally convex and C is convex. Let U and N be independent real random variables such that $EU = 1$, $P[N = n] > 0$ for $n = 1, 2, \dots$, and $\sum_{n=1}^{\infty} P[N = n] = 1$. Let $X(n, u)$ be the sequence $(u, \dots, u, 0, \dots)$ consisting of n u 's and the remainder 0 's, so $X(n, u) \in C$. Then $EX(N, U) = (q_1, q_2, \dots)$ where $q_k = P[N \geq k] > 0$, so $EX \notin C$.

We will make use of the following assumptions in Theorems 3.2-3.9:

A.1. X and Y are real, Hausdorff LTS.

A.2. C is a convex subset of X .

A.3. (Ω, \mathcal{G}, P) is a probability space.

A.4. $X: \Omega \rightarrow X$ is a Pettis integrable mapping such that $X(\Omega) \subseteq C$.

A.5. \leq_T is a component-wise ordering on \mathcal{U} determined by $T \subset \mathcal{U}^*$.

A.6. $f: C \rightarrow \mathcal{U}$ is convex with respect to \leq_T .

A.7. $f(X): \Omega \rightarrow \mathcal{U}$ is Pettis integrable.

Jensen's inequality and conditions for strict inequality are obtained first under the assumption that $C^0 \neq \emptyset$, which guarantees the existence of supporting hyperplanes. Later this assumption is replaced by the assumption that \mathcal{X} is locally convex.

Theorem 3.2.

Under assumptions A.1 - A.7, if $C^0 \neq \emptyset$, if $P[x^*(X) = \alpha] < 1$ for each supporting hyperplane $\{x | x^*(x) = \alpha\}$ of \bar{C} , and if $t(f(\cdot))$ is continuous on C^0 for each $t \in T$, then $f(EX) \leq Ef(X)$.

Proof:

Choose any $t \in T$ and set $h(x) = t(f(x))$. Since $t(Ef(X)) = Et(f(X))$ it suffices to show that $h(EX) \leq Eh(X)$. This will be seen to be a consequence of the following representation for the convex function h continuous on C^0 : for each x_0 in C^0 ,

$$(3.1) \quad h(x_0) = \max \{m(x_0) | m \text{ affine, } m \leq h \text{ on } C\},$$

where an affine function on \mathcal{X} is of the form $m(x) = x^*(x) + \alpha$ for some $x^* \in \mathcal{X}^*$ and $\alpha \in \mathbb{R}$.

To obtain (3.1) consider the subset Q of the product space $\mathbb{R} \times \mathcal{X}$ given by

$$Q = \{(\alpha, x) | \alpha \geq h(x), x \in C\}.$$

Q is a convex set since h is convex, and $Q^0 \neq \emptyset$ since h is continuous on C^0 . Fix a point $x_0 \in C^0$. Since $(h(x_0), x_0) \in \partial Q$, there exists a supporting hyperplane for Q through this point, i.e., there is a nonzero

$g \in (R \times X)^*$ and $c \in R$ such that

$$(3.2) \quad g(h(x_0), x_0) = c$$

and

$$(3.3) \quad g(\alpha, x) \geq c \quad \text{if} \quad (\alpha, x) \in Q$$

([5], Theorem 21.11 and corollary). Write $g(\alpha, x) = \alpha g(1, \varphi) + g(o, x)$; we claim that $g(1, \varphi) > 0$. If $g(1, \varphi) < 0$ then $g(\alpha, x) \rightarrow -\infty$ as $\alpha \rightarrow +\infty$, contradicting (3.3). If $g(1, \varphi) = 0$ then $g(\alpha, x) = g(o, x)$, hence $g(\alpha, x_0) = c$ for all $\alpha \in R$. Since g is nonzero there is an $x_1 \neq \varphi$ such that $g(o, x_1) = 1$, so

$$(3.4) \quad g(\alpha, x_0 - \delta x_1) = c - \delta$$

for all $\alpha, \delta \in R$. Since $x_0 \in C^0$, $\delta > 0$ can be chosen small enough so that $x_0 - \delta x_1 \in C$, so (3.4) contradicts (3.3). Therefore $g(1, \varphi) > 0$ as claimed, so we can define

$$m(x) \equiv g(1, \varphi)^{-1} [c - g(o, x)].$$

The function m is an affine function on X such that $m \leq h$ on C (by (3.3)) and $m(x_0) = h(x_0)$ (by (3.2)), which proves (3.1).

Now, to show that $h(EX) \leq Eh(X)$, by (3.1) we can choose an affine function \bar{m} such that $\bar{m} \leq h$ on C and $\bar{m}(EX) = h(EX)$ ($EX \in C^0$ by Theorem 3.1 (ii)). Since X is Pettis integrable, $\bar{m}(EX) = E \bar{m}(X) \leq Eh(X)$ as required. (The idea of this proof is similar to one of Meyer ([12], Chapter XI, p. 223) whose result is stated for compact C .) \square

If X is not a degenerate (constant) random vector and if f is strictly component-wise convex, then strict component-wise inequality holds in Jensen's inequality.

Theorem 3.3.

In addition to the hypotheses of Theorem 3.2 assume that f is strictly component-wise convex on C and that for each $x \in X$ either $A(x) \notin G$ or $P[A(x) < 1]$, where $A(x) = \{\omega | X(\omega) = x\}$. Then $f(EX) \ll Ef(X)$.

Proof:

Using the notation of the preceding proof, we must show that $h(EX) < Eh(X)$. Since h is now strictly convex and the graph of the affine function $\bar{m}(x)$ is a linear surface, $\bar{m}(x)$ and $h(x)$ can agree at only one point of C . Since $\bar{m}(EX) = h(EX)$ it follows that $\bar{m}(x) < h(x)$ if $x \in C$ and $x \neq EX$, so

$$\Omega \setminus A(EX) = \{\omega | \bar{m}(X(\omega)) < h(X(\omega))\}.$$

Therefore $A(EX) \notin G$ since \bar{m} and h are measurable, so $P[A(EX)] < 1$.

Hence $P[\bar{m}(X) < h(X)] > 0$ so $Eh(X) > E\bar{m}(X) = \bar{m}(EX) = h(EX)$. \square

To prove that strict inequality occurs in Jensen's inequality if f is strictly convex with respect to \leq_T , it must be assumed that T is total on Y and additional structure must be imposed on either X or Y . In Theorem 3.4 it is assumed that Y is a normed linear space, while in Theorem 3.5 it is assumed that X is a separable pre-Frechet space (a locally convex, metrizable LTS) and additional regularity for f is required.

Theorem 3.4.

In addition to the hypotheses of Theorem 3.2 assume that f is strictly convex with respect to \leq_T on C , that for each $x \in X$ either $A(x) \notin G$ or $P[A(x)] < 1$, that Y is a normed linear space, and that there exists a countable subset $T_0 \subset T$ such that T_0 is total on Y . Then $f(EX) < Ef(X)$.

Proof:

Let $T_0 = \{t_n\}$ and let $S_0 = \{s_n\} = \{t_n / \|t_n\|\}$ so S_0 is also total

on \mathcal{U} and each $\|s_n\| = 1$. If we define the functional s on \mathcal{U} by $s(y) = \sum_{n=1}^{\infty} 2^{-n} s_n(y)$ then $s \in \mathcal{U}^*$, and $s(y) < s(z)$ if $y < z$ since S_0 is total. Thus $h(x) = s(f(x))$ is a strictly convex real-valued function on C , so the argument used to prove Theorem 3.3 shows that $s(f(EX)) = h(EX) < Eh(X) = s(Ef(X))$. Thus $f(EX) \neq Ef(X)$, so $f(EX) < Ef(X)$. \square

Remark 3.3.

If T is total on \mathcal{U} and if \mathcal{U}^* is separable in the strong (norm) or weak* topology then the last assumption in Theorem 3.4 is necessarily satisfied (e.g. $\mathcal{U} = L_p$, $1 < p < \infty$, or \mathcal{U} a separable Hilbert space). This remark also applies to Theorem 3.8.

Theorem 3.5.

In addition to the hypotheses of Theorem 3.2 assume that f is strictly convex with respect to \leq_T on C , that for each $x \in \mathcal{X}$ either $A(x) \notin \mathcal{G}$ or $P[A(x)] < 1$, that \mathcal{X} is a separable metrizable locally convex LTS, that $t(f(\cdot))$ is lower semicontinuous on C for each $t \in T$, and that T is total on \mathcal{U} . Then $f(EX) < Ef(X)$.

Proof:

First, because of the additional structure assumed for \mathcal{X} , the weakly measurable mapping X is in fact a Borel measurable mapping from (Ω, \mathcal{G}) to $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. This is proved, for example, by Ahmad ([1], Corollary to Proposition 1, p. 100) and is an extension a well-known result of Pettis [14] for a normed linear space (see also [9], Theorem 3.5.3). In particular $A(x) \in \mathcal{G}$ for each x .

Let \mathcal{U} be a countable collection of open sets forming a base for the topology of \mathcal{X} . There must exist some $x_0 \in (C \setminus \{EX\})$ such that $P[X \in U] > 0$ for every open neighborhood U of x_0 . For if not, then for each $x \in (C \setminus \{EX\})$

there exists some $U(x) \in \mathcal{U}$ such that $x \in U(x)$ and $P[X \in U(x)] = 0$, implying that the event $\{X \neq EX\}$ can be expressed as a countable union of null events, so $P[X = EX] = 1$, a contradiction. Next, since f is strictly convex and T is total on \mathcal{U} there exists $t \in T$ such that

$$h\left(\frac{1}{2}x_0 + \frac{1}{2}EX\right) < \frac{1}{2}h(x_0) + \frac{1}{2}h(EX)$$

where $h(x) = t(f(x))$. Therefore $\bar{m}(x_0) < h(x_0)$ where $\bar{m}(x)$ is the affine function obtained in the proof of Theorem 3.2 which supports h from below and for which $\bar{m}(EX) = h(EX)$. Since h is lower semicontinuous and $\bar{m}(x)$ is continuous on C , the set $U_0 = \{x | x \in C, \bar{m}(x) < h(x)\}$ is relatively open in C and contains x_0 , so $P[X \in U_0] > 0$. Therefore $Eh(X) > E\bar{m}(X) = h(EX)$, implying $t(Ef(X)) > t(f(EX))$, so $f(EX) < Ef(X)$. \square

Remark 3.4.

The assumption that \mathcal{X} is locally convex is imposed in Theorem 3.5 only to guarantee (along with separability and metrizability) that the weakly measurable mapping X is in fact Borel measurable, and thus can be dropped if this latter fact is assumed.

Up to now the assumption $C^0 \neq \emptyset$ has been used to guarantee the existence of supporting hyperplanes for the set $Q = \{(\alpha, x) | \alpha \geq h(x), x \in C\}$ defined in the proof of Theorem 3.2. For the rest of this section we drop this assumption and instead assume that \mathcal{X} is locally convex and C is closed. Jensen's inequality is now obtained by using the fact that in a locally convex space two closed convex disjoint sets, one of which is compact, can be strictly separated by a closed hyperplane ([5], Theorem 21.12).

Theorem 3.6.

Under assumptions A.1-A.7, if C is closed, if \mathcal{X} is locally convex, and if $t(f(\cdot))$ is lower semicontinuous on C for each $t \in T$, then

$$f(EX) \leq Ef(X).$$

Proof:

Choose $t \in T$ and let $h(x) = t(f(x))$. As a consequence of the strict separation property mentioned above it can be shown by an argument similar to that leading to (3.1) that for each x in C ,

$$h(x) = \sup\{m(x) \mid m \text{ affine, } m < h \text{ on } C\}$$

(c.f. [5], Proposition 21.18). For any such m , $m(EX) = Em(X) < Eh(X)$ so $h(EX) = \sup_{m < h} m(EX) \leq Eh(X)$, completing the proof (note that $EX \in C$ by Theorem 3.1 (i)). \square

The necessity of the assumption of lower semicontinuity in Theorem 3.6 is illustrated by the example in the third paragraph of section 1, involving Fatou's Lemma. In the example $\mathcal{X} \equiv \mathbb{R}^\infty$ and the dual space \mathcal{X}^* consists of all linear functionals x^* of the form

$$x^*(x_1, x_2, \dots) = \sum_{i=1}^r \alpha_i x_i$$

for some $r \geq 1$ and $\alpha_i \in \mathbb{R}$. Thus $X = (X_1, X_2, \dots)$ is Pettis integrable if and only if each X_n is integrable, and

$$EX = (EX_1, EX_2, \dots).$$

If we take $\mathcal{U} = \mathbb{R}$, let \leq be the natural ordering on \mathbb{R} , and assume $E(\limsup X_n) < \infty$ (otherwise (1.4) is trivially satisfied), then assumptions A.1-A.7 are satisfied, where f is given by (1.3). Furthermore, C is closed and $\mathcal{X} = \mathbb{R}^\infty$ is locally convex, so all assumptions of Theorem 3.6 are satisfied except that f is not lower semicontinuous on C (this amounts to the fact that the value of a double limit may change if the order of the two limit operations is reversed). However, as pointed out in section 1, the conclusion of Theorem 3.6, i.e. inequality (1.4), fails.

Results concerning strict inequality, analogous to Theorems 3.3, 3.4, and 3.5, are now presented. The inner measure induced by P is denoted by P_* .

Theorem 3.7.

In addition to the hypotheses of Theorem 3.6 assume that f is strictly component-wise convex with respect to \leq_T on C and that $P_*[A(x)] < 1$ for each $x \in X$. Then $f(EX) < Ef(X)$.

Proof:

Using the notation of the preceding proof, there exists a sequence $\{m_n\}$ of affine functions such that $m_n < h$ on C and $m_n(EX) \rightarrow h(EX)$. If $h(EX) = Eh(X)$ then $E|h(X) - m_n(X)| \rightarrow 0$, so there exists a subsequence $\{m_k\} \subseteq \{m_n\}$ and a subset $\Omega_0 \in \mathcal{G}$ such that $m_k(X(\omega)) \rightarrow h(X(\omega))$ for each $\omega \in \Omega_0$ and such that $P(\Omega_0) = 1$. Now, $X(\Omega_0)$ must contain at least two distinct points, say x_1 and x_2 , (otherwise, if $X(\Omega_0) = \{x\}$ alone, then $P_*[A(x)] = 1$) so $m_k(x_i) \rightarrow h(x_i)$, $i = 1, 2$. Therefore

$$m_k\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) \rightarrow \frac{1}{2}h(x_1) + \frac{1}{2}h(x_2) > h\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right)$$

(h is strictly convex). Since $m_k < h$ this is a contradiction, so it must be that $h(EX) < Eh(X)$. \square

Remark 3.5.

The assumption that $P_*[A(x)] < 1$ for each x is slightly stronger than the assumption in Theorems 3.3-3.5 that either $A(x) \notin \mathcal{G}$ or $P[A(x)] < 1$. If (Ω, \mathcal{G}, P) is a complete probability space, however, the two assumptions are equivalent.

To obtain strict inequality when f is strictly convex we again have to impose further regularity and countability restrictions.

Theorem 3.8.

In addition to the hypotheses of Theorem 3.6 assume that f is strictly convex with respect to \leq_T on C , that $P_*[A(x)] < 1$ for each $x \in X$, that \mathcal{U} is a normed linear space, and that there exists a countable subset $T_0 \subset T$ such that T_0 is total on \mathcal{U} . Then $f(EX) < Ef(X)$.

Proof:

Use the proof of Theorem 3.4 with "Theorem 3.7" substituted for "Theorem 3.3". \square

The next theorem is quite similar to Theorem 3.5, differing mainly in that C is assumed closed rather than $C^\circ \neq \emptyset$.

Theorem 3.9.

In addition to the hypotheses of Theorem 3.6 assume that f is strictly convex with respect to \leq_T on C , that $P_*[A(x)] < 1$ for each $x \in X$, that X is a separable metrizable locally convex LTS, that $t(f(\cdot))$ is continuous on C for each $t \in T$, and that T is total on \mathcal{U} . Then $f(EX) < Ef(X)$.

Proof:

As pointed out in the proof of Theorem 3.5, X is Borel measurable. Let x_0 and h be as defined in the proof of Theorem 3.5, and suppose $h(EX) = Eh(X)$. As in the proof of Theorem 3.7, there exists a sequence $\{m_k\}$ of affine functions and a subset $\Omega_0 \in \mathcal{G}$ such that $m_k(EX) \rightarrow h(EX)$, $m_k < h$ on C , $m_k(X(\omega)) \rightarrow h(X(\omega))$ for each $\omega \in \Omega_0$, and $P(\Omega_0) = 1$. Let $\{U_n\}$ be a decreasing sequence of open neighborhoods of x_0 such that $\bigcap U_n = \{x_0\}$, let ω_n be any element of $X^{-1}(U_n) \cap \Omega_0$ (which is nonempty since $P[X \in U_n] > 0$), and let $x_n = X(\omega_n)$. Then $x_n \rightarrow x_0$ and $\lim_{k \rightarrow \infty} m_k(x_n) = h(x_n)$ for each n . Since h is continuous on C and

$$h\left(\frac{1}{2}x_0 + \frac{1}{2}EX\right) < \frac{1}{2}h(x_0) + \frac{1}{2}h(EX),$$

there exists n' such that

$$h\left(\frac{1}{2}x_{n'} + \frac{1}{2}EX\right) < \frac{1}{2}h(x_{n'}) + \frac{1}{2}h(EX).$$

But this is impossible since the affine functions m_k converge to h at each of the three points $x_{n'}$, EX , and $\frac{1}{2}x_{n'} + \frac{1}{2}EX$. Thus $h(EX) < Eh(X)$, so $f(EX) \neq Ef(X)$, hence $f(EX) < Ef(X)$. \square

Conditions under which Jensen's inequality holds for a convex real-valued function f defined on a LTS \mathcal{X} can be obtained simply by taking $\mathcal{U} = R$ in Theorems 3.2 and 3.6. The following theorem extends these results to the important case of a function which may assume infinite values.

Theorem 3.10.

Let A.1-A.4 be satisfied and let $f = C \rightarrow [-\infty, \infty)$ be a convex extended-real-valued function such that $Ef(X)$ exists. If either

(i) $C^0 \neq \emptyset$, $P[x^*(X) = \alpha] < 1$ for each supporting hyperplane $\{x | x^*(x) = \alpha\}$ of \bar{C} , f continuous on C^0 , or,

(ii) C closed, \mathcal{X} locally convex, f lower semicontinuous on C ,

then $f(EX) \leq Ef(X)$.

Proof:

If $Ef(X) = +\infty$ the desired inequality is trivial so assume $Ef(X) < \infty$. Under condition (i) (respectively, condition (ii)) for each $\alpha \in R$ the function $\max(f(x), \alpha)$ is also convex and continuous on C^0 (respectively, lower semicontinuous on C), so

$$\max(f(EX), \alpha) \leq E \max(f(X), \alpha)$$

by Theorem 3.2 (respectively, Theorem 3.6). The desired inequality follows by letting $\alpha \rightarrow -\infty$ and applying the Monotone Convergence Theorem. \square

The results in this section have been obtained under the assumption that X is Pettis integrable. Some recent results concerning the existence of the Pettis integral (\equiv barycenter) are given by Bourgin [4] and Khurana [10]. In the next section the stronger assumption that X is Bochner integrable is imposed, necessitating consideration of LTS \mathcal{X} which are metrizable and complete.

4. Jensen's inequality for Bochner integrable functions.

In the preceding section the supporting and separating hyperplane theorems were used to obtain Jensen's inequality for Pettis integrable functions convex with respect to a component-wise ordering on \mathcal{U} . When the stronger assumption of Bochner integrability is imposed, however, the Strong Law of Large Numbers (SLLN) easily yields a generalized Jensen's inequality for a wider class of functions, those convex with respect to a more general binary relation on \mathcal{U} , not necessarily a partial ordering. Since a limiting process is used, this method does not apply to questions of strict inequality.

We now impose seven assumptions, corresponding to A.1-A.7 in section 3:

- B.1. \mathcal{X} and \mathcal{U} are real separable Banach spaces.
- B.2. C is a convex subset of \mathcal{X} .
- B.3. (Ω, \mathcal{G}, P) is a probability space.
- B.4. $X: \Omega \rightarrow \mathcal{X}$ is a Bochner integrable mapping such that $X(\Omega) \subseteq C$.
- B.5. W is a subset of the product space $\mathcal{U} \times \mathcal{U}$.
- B.6. $f: C \rightarrow \mathcal{U}$ is W -convex on C (defined below).
- B.7. $f(X): \Omega \rightarrow \mathcal{U}$ is Bochner integrable.

A mapping $X: \Omega \rightarrow \mathcal{X}$ is Bochner integrable with respect to (Ω, \mathcal{G}, P) if (a) X is Borel (\equiv weakly) measurable and (b) $\|X\|$ is integrable (see [9], Chapter III). The Bochner integral is denoted by EX ; if it exists then so does the Pettis integral and the two are identical.

An arbitrary subset W of the product space $\mathcal{Y} \times \mathcal{Y}$ determines a binary relation $\overset{W}{\sim}$ defined as follows: $y \overset{W}{\sim} z \Leftrightarrow (y, z) \in W$. A function $f: C \rightarrow \mathcal{Y}$ is W-convex on C if

$$f\left(\sum_{k=1}^n \alpha_k x_k\right) \overset{W}{\sim} \sum_{k=1}^n \alpha_k f(x_k)$$

for all integers $n \geq 2$ whenever each $x_k \in C$ and $\alpha_k \geq 0$ with $\sum \alpha_k = 1$. (Here we cannot deduce this relation for $n > 2$ from the case $n = 2$.)

The function f is W-continuous at a point $x \in C$ if the following condition is satisfied: if $\{x_n\} \subset C$ and $\{z_n\} \subset \mathcal{Y}$ are sequences such that $x_n \rightarrow x$, $z_n \rightarrow z$, and $(f(x_n), z_n) \in W$ for each n , then $(f(x), z) \in W$. (For non-metrizable \mathcal{Y} we would replace sequences by nets.)

Since Bochner integrability implies Pettis integrability, Theorem 3.1 provides conditions which guarantee that $EX \in C$; these will not be repeated here. The main result concerning W-convex Bochner integrable functions is the following:

Theorem 4.1.

Under assumptions B.1-B.7, if $EX \in C$ and if f is W-continuous at EX then $f(EX) \overset{W}{\sim} Ef(X)$.

Proof:

Let X_1, X_2, \dots be a sequence of independent \mathcal{X} -valued random vectors, each distributed according to the probability law of X . Since f is W-convex,

$$f\left(\frac{1}{n} \sum_{k=1}^n X_k\right) \overset{W}{\sim} \frac{1}{n} \sum_{k=1}^n f(X_k)$$

for each n . By the SLLN for Bochner-integrable random vectors ([2], [8], [13])

$$\frac{1}{n} \sum_{k=1}^n X_k \rightarrow EX \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^n f(X_k) \rightarrow Ef(X) \quad \text{a.s.}$$

Since $EX \in C$ and f is W -continuous at EX , this implies that

$$f(EX) \overset{W}{\sim} Ef(X). \quad \square$$

Remark 4.1.

If f is continuous at x and if W is closed (in the product topology) then f is W -continuous at x . Notice that W is closed iff $\overset{W}{\sim}$ is a closed binary relation in the sense of condition (ii) of the definition of a closed cone ordering (section 2).

Remark 4.2.

If $\overset{W}{\sim}$ is actually a component-wise ordering \leq_T and if $t(f(x))$ is lower semicontinuous at x for each $t \in T$ then f is W -continuous at x . Therefore the continuity conditions assumed in Theorems 3.2-3.10 are special cases of W -continuity.

Remark 4.3.

The Bochner integral can be defined for mappings assuming values in a separable Frechet space (complete metrizable locally convex LTS) and the SLLN continues to hold (Ahmad [1], Proposition 2, p. 114). Thus B.1 can be weakened to the assumption that X and Y are separable Frechet spaces.

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